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# On Multiple Integrals of Special Form

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**Abstract**—Multiple integrals generalizing the iterated kernels of integral operators are expressed as single integrals in the case of a special representation of the kernel (this is our theorem). Besides integral equations, Markov processes involve these integrals as well. As a consequence of the theorem, we obtain transition probability densities of certain Markov processes. As an illustration, we consider nine examples.

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Few nontrivial multiple integrals expressible in terms of simple analytic formulas are known. In our previous paper [1], we increased their number by evaluating a $(2m-4)$-fold multiple integral depending on $(m+3)$ parameters; this integral occurred in a problem concerning the zeros of Gaussian stationary Markov processes [2]. Here $m$ is an integer, $m>2$. However, as it turned out, the method used in [1] is not closely tied to the specific problem examined there; it can be extended to other multiple integrals with a view to obtain their representation in simple analytic form. The proof of the main result (the theorem) is followed by examples in which we specialize it to cover concrete integrand functions related to certain classical integral transforms (Fourier, Hankel, and Kontorovich–Lebedev). As a consequence of the theorem, we obtain an expression for the transition probability density of certain (hitherto unknown) Markov processes.

Consider the integral

$$\pi_{m-1}(x_m|a_{m-1},\dots,a_1|x_1)=\int_\Omega\cdots\int_\Omega\prod_{k=1}^{m-1}\pi_1(x_{k+1}|a_k|x_k)\,dx_2\dots dx_{m-1}, \tag{1}$$

in which $\Omega$ is an interval (possibly, infinite) of the real axis, $x_1,\dots,x_m$ are real variables, and $a_1,\dots,a_{m-1}$ are parameters (possibly, multidimensional). In particular, when all the $a_k$ are equal, $a_k=a$, the integral (1) is the iterated kernel of the integral operator

$$\int_\Omega \pi_1(y|a|x)f(x)\,dx. \tag{2}$$

In [1], the integral (1) was to reduced a single one for $\Omega=[0,\infty)$, $a_k=(\mu,\lambda,b_k)$, $\operatorname{Re}\mu>0$, $|b_k|\le 1$, $k=1,\dots,m-1$,

$$\pi_1(y|a_k|x)=y^{\mu-1}Y(\mu,\lambda;A_k(x,y))x^{\mu-2},\qquad Y(\mu,\lambda;x)=2\left(\frac{x}{2\lambda}\right)^{(1-\mu)/2}K_{1-\mu}(\sqrt{2\lambda}x),$$

and $A_k(x,y)=x^2+2b_kxy+y^2$, where $K_\nu(x)$ is the modified Bessel function (the MacDonald function) [3].

By analogy with (2), the function $\pi_1(y|a_k|x)$ in (1) is called a *kernel*.

Suppose that, in (1), the kernel can be expressed as

$$\pi_1(y|a|x)=\int_\Lambda\psi_\lambda(a)u_\lambda(y)v_{f(\lambda)}(x)\,d\lambda \tag{3}$$

or

$$\pi_1(y|a|x) = \int_\Lambda \psi_\lambda(a) u_{g(\lambda)}(y) v_\lambda(x)\, d\lambda, \tag{4}$$

where $\Lambda$ is an interval (possibly, infinite) of the real axis, $\psi_\lambda(a)$ is a function of a real variable $\lambda$ and a parameter $a$, while the functions $u_\lambda(x)$ and $v_\lambda(x)$ satisfy

$$\int_\Omega u_{\lambda_1}(x) v_{\lambda_2}(x)\, dx = \delta(\lambda_1 - \lambda_2). \tag{5}$$

Here $\delta(\lambda)$ is the $\delta$-function, $f(\lambda) \in \Lambda$, and $g(\lambda) \in \Lambda$ are real functions.

Classical integral transformations. are based on relation (5). For example, [4], [5]:

- for the Fourier cosine transform,

$$\Omega = [0, \infty), \qquad u_\lambda(x) = v_\lambda(x) = \sqrt{\frac{2}{\pi}} \cos \lambda x;$$

- for the Fourier sine transform,

$$\Omega = [0, \infty), \qquad u_\lambda(x) = v_\lambda(x) = \sqrt{\frac{2}{\pi}} \sin \lambda x;$$

- for the Fourier exponential transform,

$$\Omega = (-\infty, \infty), \qquad u_\lambda(x) = \frac{1}{\sqrt{2\pi}}\, e^{-i\lambda x}, \quad v_\lambda(x) = \frac{1}{\sqrt{2\pi}}\, e^{i\lambda x},$$

- for the Hankel transform,

$$\Omega = [0, \infty), \qquad u_\lambda(x) = v_\lambda(x) = \sqrt{\lambda x}\, J_\nu(\lambda x), \quad \operatorname{Re} \nu > -\frac{1}{2},$$

- for the Kontorovich–Lebedev transform,

$$\Omega = [0, \infty), \qquad u_\lambda(x) = \frac{2}{\pi^2}\, \lambda \cdot \sinh(\pi\lambda) \cdot \frac{K_{i\lambda}(x)}{x}, \quad v_\lambda(x) = K_{i\lambda}(x).$$

Here $J_\nu(x)$ is the Bessel function of the first kind [3], [5], $K_\nu(x)$ is the MacDonald function [3], [5], and $i$ is the imaginary unit.

Denote by

$$f_n(\lambda) = \underbrace{f(f(f(\cdots(f}_{n \text{ times}}(\lambda))))) \tag{6}$$

the $n$th iteration of the function $f(\lambda)$. Similarly, $g_n(\lambda)$ denotes the $n$th iteration of the function $g(\lambda)$.

**Theorem.** *The integral* (1) *with kernel of the form* (3) *can be reduced to the single integral*

$$\int_\Lambda \left[ \prod_{k=1}^{m-2} \psi_{f_k(\lambda)}(a_{m-k-1}) \right] \psi_\lambda(a_m - 1) u_\lambda(x_m) v_{f_{m-1}(\lambda)}(x_1)\, d\lambda, \tag{7}$$

*and when it has a kernel of the form* (4), *to the single integral*

$$\int_\Lambda \left[ \prod_{k=1}^{m-2} \psi_{g_k(\lambda)}(a_{k+1}) \right] \psi_\lambda(a_1) u_{g_{m-1}(\lambda)}(x_m) v_\lambda(x_1)\, d\lambda. \tag{8}$$

*In the case $f(\lambda) = \lambda$, the integral* (1) *takes the following especially simple form*:

$$\int_\Lambda \left[ \prod_{k=1}^{m-1} \psi_\lambda(a_k) \right] u_\lambda(x_m) v_\lambda(x_1)\, d\lambda. \tag{9}$$

**Proof.** Let us use the recurrence relation

$$\pi_{m-1}(x_m|a_{m-1},\dots,a_1|x_1) = \int_\Omega \pi_{m-2}(x_m|a_{m-1},\dots,a_2|x_2)\pi_1(x_2|a_1|x_1)\,dx_2. \tag{10}$$

For $m=3$, relation (10) implies

$$\pi_2(x_3|a_2,a_1|x_1) = \int_\Omega \pi_1(x_3|a_2|x_2)\pi_1(x_2|a_1|x_1)\,dx_2. \tag{11}$$

In (11), substituting expression (3) for $\pi_1$ and rearranging the integration over $\lambda_1$, $\lambda_2$ and $x_2$ in the resulting triple integral, we obtain

$$\pi_2(x_3|a_2,a_1|x_1) = \int_\Lambda\int_\Lambda \psi_{\lambda_2}(a_2)\psi_{\lambda_1}(a_1)u_{\lambda_2}(x_3)v_{f(\lambda_1)}(x_1)K(\lambda_1,\lambda_2)\,d\lambda_1 d\lambda_2, \tag{12}$$

where

$$K(\lambda_1,\lambda_2) = \int_\Omega u_{\lambda_1}(x_2)v_{f(\lambda_2)}(x_2)\,dx_2. \tag{13}$$

By (5), the integral (13) is equal to $\delta(\lambda_1 - f(\lambda_2))$, so that, for $f(\lambda)\in\Lambda$, only one integral over $\lambda_2$ remains in (12):

$$\pi_2(x_3|a_2,a_1|x_1) = \int_\Lambda \psi_{\lambda_2}(a_2)\psi_{f(\lambda_2)}(a_1)u_{\lambda_2}(x_3)v_{f_2(\lambda_2)}(x_1)\,d\lambda_2. \tag{14}$$

This integral coincides with (7) for $m=3$ and $\lambda_2=\lambda$. Formula (7) is now obtained by induction on $m\ge 3$, the induction base being formula (14).

Similarly, we can prove formula (8), which is the case in which the kernel is expressed in the form (4). □

We do not draw the reader's attention to the justification of the operations with $\delta$-functions, because they have now been in use for quite some time. It should only be kept in mind that $\delta(x)$ is the "kernel" of a linear functional assigning to functions from a particular function space $\mathscr{F}$ (compactly supported functions which are continuous at zero are mostly considered) its value at $x=0$. Accordingly, the functions $\psi_\lambda(a)u_\lambda(x)$ and $\psi_\lambda(a)v_\lambda(y)$ appearing in the integrals (3), (4) and regarded as functions of the variable $\lambda$ must belong to the space $\mathscr{F}$.

An interesting modification of the theorem is obtained when, in (1), we put

$$a_j = (t_{j+1},t_j), \qquad j=1,\dots,m-1, \quad t_j<t_{j+1}$$

and, instead of (3), we have the following representation for the kernel:

$$\pi_1(y|(t,s)|x) = \int_\Lambda \frac{\psi_\lambda(t)}{\psi_\lambda(s)}\,u_\lambda(y)v_\lambda(x)\,d\lambda, \tag{15}$$

in which $s<t$, $\psi_\lambda(t)$ is a real function of the argument $\lambda$ and the parameter $t$, while, as before, $u_\lambda(x)$ and $v_\lambda(x)$ satisfy relation (5).

Substituting (15) into (12)–(14), we see that

$$\pi_2(x_3|(t_3,t_2),(t_2,t_1)|x_1) = \pi_1(x_3|(t_3,t_1)|x_1),$$

and, from (12), we obtain the equation

$$\pi_1(x_3|(t_3,t_1)|x_1) = \int_\Omega \pi_1(x_3|(t_3,t_2)|x_2)\pi_1(x_2|(t_2,t_1)|x_1)\,dx_2. \tag{16}$$

In view of the recurrence relation (10), this equation implies the equality

$$\pi_{m-1}(x_3|(t_m,t_{m-1}),\dots,(t_2,t_1)|x_1) = \pi_1(x_m|(t_m,t_1)|x_1). \tag{17}$$

If

$$\pi_1(y|(t,s)|x)\ge 0, \qquad \int_\Omega \pi_1(y|(t,s)|x)\,dy \le 1, \qquad s<t, \tag{18}$$

then we can identify

$$\pi_1(y|(t,s)|x) = \pi_{s\to t}(x \to y) \tag{19}$$

with the transition probability density of the trajectory of some Markov process with continuous time from the initial point $x$ at time $s$ to the neighborhood of the point $y$ at time $t$, because, in this case, Eq. (16)

$$\pi_{t_1\to t_3}(x_1 \to x_3) = \int_\Omega \pi_{t_1\to t_2}(x_1 \to x_2)\pi_{t_2\to t_3}(x_2 \to x_3)\,dx_2, \tag{20}$$

coincides with the Chapman–Kolmogorov equation. Thus, we have the following statement.

**Corollary.** *If, in the kernel $\pi_1(y|a_j|x)$ with parameters $a_j = (t_{j+1}, t_j)$, $t_1 < t_2 < \dots < t_{m+1}$, the kernel can be expressed as* (15)*, then the integral* (1) *can be reduced to the form* (17)*. If, moreover, relations* (18) *hold, then, by formula* (19)*, this kernel can be identified with the transition probability density of some Markov process with continuous time $t$.*

Let us pass to examples, using each of the classical integral transforms mentioned above.

**Example 1.** Suppose that $\Omega = [0,\infty)$ and the kernel is of the form

$$\pi_1(y|a_j|x) = \frac{1}{\pi}\left[\frac{|a_j|}{a_j^2 + (y+x)^2} + \frac{|a_j|}{a_j^2 + (y-x)^2}\right], \qquad x > 0, \quad y > 0. \tag{21}$$

Let us apply to (21) the Fourier cosine transform [5, Vol. 1]:

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \pi_1(y|a_j|x)\cos\lambda x\,dx = \sqrt{\frac{2}{\pi}}\exp\{-|a_j|\lambda\}\cos\lambda y. \tag{22}$$

Solving Eq. (22) for $\pi_1(y|a_j|x)$, i.e., again applying to it the cosine transform, we find

$$\pi_1(y|a_j|x) = \frac{2}{\pi}\int_0^\infty \exp\{-|a_j|\lambda\}\cos\lambda y\cos\lambda x\,d\lambda. \tag{23}$$

Therefore, the kernel (21) can be expressed as (3), (5) with

$$f(\lambda) = \lambda, \qquad \Lambda = [0,\infty), \qquad \psi_\lambda(a_j) = e^{-|a_j|\lambda}, \quad u_\lambda(x) = v_\lambda(x) = \sqrt{\frac{2}{\pi}}\cos\lambda x$$

and, therefore, by the theorem, we have

$$\begin{aligned}\pi_{m-1}(x_m|a_{m-1},\dots,a_1|x_1) &= \frac{2}{\pi}\int_0^\infty \exp\left\{-\lambda\sum_{j=1}^{m-1}|a_j|\right\}\cos\lambda y\cos\lambda x\,d\lambda \\ &= \pi_1\left(x_m\,\middle|\,\sum_{j=1}^{m-1}|a_j|\,\middle|\,x_1\right).\end{aligned} \tag{24}$$

If we set $a_j = t_{j+1} - t_j$, $t_1 < t_2 < \dots < t_m$, then we can identify (24) with the transition probability density of the Markov process with continuous time $t$ and with values on the semiaxis $\Omega = [0,\infty)$:

$$\pi_1(y|t_{j+1} - t_j|x) = \pi_{t_j\to t_{j+1}}(x \to y), \tag{25}$$

because $\pi_1(y|t_{j+1} - t_j|x) \ge 0$ and

$$\int_0^\infty \pi_1(y|t_{j+1} - t_j|x)\,dy = 1,$$

i.e., conditions (18) hold, and, by (24), the Chapman–Kolmogorov equation (20) is satisfied. From (24), we see that

$$\psi_\lambda(t-s) = e^{-\lambda(t-s)} = \frac{\psi_\lambda(t)}{\psi_\lambda(s)},$$

so that this example also illustrates the corollary.

Now, a few words about this Markov process. Its transition probability density (25):

- is stationary (i.e., does not depend on the initial time):

$$\pi_{s\to t}(x \to y) = p_{t-s}(x, y) \equiv p;$$

- is symmetric with respect to the initial and finite values of the process:

$$\pi_{s\to t}(x \to y) = \pi_{s\to t}(y \to x);$$

- is a harmonic function in the variables $t$, $x$ and $t$, $y$:

$$\frac{\partial^2 p}{\partial t^2} + \frac{\partial^2 p}{\partial x^2} = 0, \qquad \frac{\partial^2 p}{\partial t^2} + \frac{\partial^2 p}{\partial y^2} = 0;$$

- satisfies the hyperbolic equation in the variables $x$, $y$:

$$\frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p}{\partial y^2}.$$

It follows from (23) that

$$\lim_{s\to t} \pi_{s\to t}(x \to y) = \delta(y - x),$$

i.e., the trajectory of the process is not discontinuous.

**Example 2.** Suppose that $\Omega = [0, \infty)$, and the kernel is of the form [3, p. 436, no. 3.762(3)]

$$\begin{aligned} \pi_1(y|a|x) &= \frac{2}{\pi} \int_0^\infty \lambda^{-a} \cos \lambda y \cos \lambda x \, d\lambda \\ &= \frac{\Gamma(1-a)}{\pi} \cos \frac{(1-a)\pi}{2} [(x+y)^{a-1} + |x-y|^{a-1}], \end{aligned} \tag{26}$$

where $0 < \operatorname{Re} a < 1$. Thus, the kernel (26) can be expressed as (3), (5) with

$$f(\lambda) = \lambda, \quad \Lambda = [0, \infty), \quad \psi_\lambda(a) = \lambda^{-a}, \quad u_\lambda(x) = v_\lambda(x) = \sqrt{2/\pi} \cos \lambda x$$

and, therefore, by the theorem (formula (9)), we have

$$\pi_{m-1}(x_m|a_{m-1}, \dots, a_1|x_1) = \frac{2}{\pi} \int_0^\infty \biggl[\prod_{j=1}^{m-1} \lambda^{-a_j}\biggr] \cos \lambda y \cos \lambda x \, d\lambda = \pi_1\biggl(x_m \biggm| \sum_{j=1}^{m-1} a_j \biggm| x_1\biggr);$$

moreover, the following inequalities must hold:

$$0 < \operatorname{Re} \sum_{j=1}^{m-1} a_j < 1, \qquad x_m \neq x_1$$

for the integral (1) to converge. If we set

$$a_j = t_{j+1} - t_j, \qquad 0 < t_1 < \dots < t_m,$$

then Eq. (26) yields the Chapman–Kolmogorov equation

$$\pi_1(x_3|t_3 - t_1|x_1) = \int_0^\infty \pi_1(x_2|t_2 - t_1|x_1) \pi_1(x_3|t_3 - t_2|x_2) \, dx_2,$$

but we cannot identify the kernel $\pi_1(y|t-s|x)$ with the transition probability density for some Markov process, because this kernel is not integrable over $y$ in $[0, \infty)$, and hence the second of conditions (18) does not hold.

**Example 3.** By [3, p. 494, no. 3.898(2)] the kernel

$$\pi_1(y|a|x) = \frac{1}{2\sqrt{\pi a}}\left\{\exp\left[-\frac{(x-y)^2}{4a}\right] + \exp\left[-\frac{(x+y)^2}{4a}\right]\right\} \tag{27}$$

can be expressed as (3), (5), where $\operatorname{Re} a > 0$,

$$f(\lambda) = \lambda, \quad \psi_\lambda(a) = \exp(-a\lambda^2), \quad \Lambda = \Omega = [0, \infty), \quad u_\lambda(x) = v_\lambda(x) = \sqrt{\frac{2}{\pi}}\cos \lambda x.$$

Namely,

$$\pi_1(y|a|x) = \frac{2}{\pi}\int_0^\infty \exp(-a\lambda^2)\cos\lambda y \cos\lambda x \, d\lambda.$$

Take $a_j = t_{j+1} - t_j$, $t_1 < t_2 < \dots < t_m$; then we see that

$$\psi_\lambda(a_j) = \exp(-(t_{j+1} - t_j)\lambda^2) = \frac{\psi_\lambda(t_{j+1})}{\psi_\lambda(t_j)}\,,$$

i.e., the representation (15) is valid and so is the corollary of the theorem. We also have

$$\int_0^\infty \pi_1(y|a|x)\,dy = 2 - \Phi\left(\frac{-x}{\sqrt{2a}}\right) - \Phi\left(\frac{x}{\sqrt{2a}}\right) = 1, \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x \exp\left(-\frac{y^2}{2}\right)dy. \tag{28}$$

By (28), inequalities (18) hold with $a = t - s > 0$. Thus, by formula (19), we can identify the kernel (27) with the transition probability density of some Markov process. The integral (1) itself can be calculated by using (24).

In the following two examples, $f(\lambda) \neq \lambda$.

**Example 4.** For $x > 0$, $y > 0$, by [3, p. 409, no. 3.697(7)], we have

$$\pi_1(y|x) = \frac{1}{\sqrt{2\pi x}}\left(\cos\frac{y}{4x} + \sin\frac{y}{4x}\right) = \frac{2}{\pi}\int_0^\infty \cos x\lambda^2 \cos y\lambda\, d\lambda.$$

As we see, this kernel can be expressed as (3) in which there is no parameter $a$, while

$$f(\lambda) = \lambda^2, \quad \psi_\lambda(a) = 1, \quad \Omega = \Lambda = [0, \infty), \quad u_\lambda(x) = v_\lambda(x) = \cos\lambda x,$$

so that, by the theorem, the integral (1) can be reduced to the single integral (7) in which, by (6), we have $f_k(\lambda) = \lambda^{2^k}$.

In the following example, integration is performed over a finite interval in the integral representation of the kernel $\Lambda = [0, \pi]$, i.e., in relation (3).

**Example 5.** By [3, p. 415, no. 3.715(8)], we have

$$\pi_1(y|x) = -\frac{2y}{\pi}\sin y\pi \cdot s_{-1,y}(x) = \frac{2}{\pi}\int_0^\pi \cos(x\sin\lambda)\cos y\lambda\, d\lambda,$$

where $s_{\mu,\nu}(x)$ is the Lommel function [3]. Thus, this kernel can be written as (3) in which there is no parameter $a$, while

$$\psi_\lambda(a) = 1, \quad f(\lambda) = \sin\lambda, \quad \Lambda = [0, \pi], \quad \Omega = [0, \infty), \quad u_\lambda(x) = v_\lambda(x) = \cos\lambda x.$$

The assumptions of the theorem are satisfied and the $(m-2)$-fold multiple integral (1) can be expressed as the single integral (7); moreover, by (6), we have

$$f_k(\lambda) = \underbrace{\sin(\sin(\sin(\cdots(\sin}_{k \text{ times}}\lambda)))).$$

Finally, let us present examples in which we take the kernels of classical integral transforms other than the Fourier cosine transform for $u_\lambda(x)$ and $v_\lambda(x)$.

**Example 6** (with the Fourier sine transform)**.** Suppose that, in (1), $\Omega = [0, \infty)$ and the kernel is of the form

$$\pi_1(y|a_j|x) = \frac{1}{\pi}\left[\frac{|a_j|}{a_j^2 + (y-x)^2} - \frac{|a_j|}{a_j^2 + (y+x)^2}\right], \qquad x > 0, \quad y > 0. \tag{29}$$

Twice applying the Fourier sine transform to (29), we find [5]

$$\pi_1(y|a_j|x) = \frac{2}{\pi}\int_0^\infty \exp\{-|a_j|\lambda\} \sin \lambda y \sin \lambda x \, d\lambda.$$

Therefore, the kernel is of the form (3), with

$$f(\lambda) = \lambda, \quad \Lambda = [0, \infty), \quad \psi_\lambda(a) = \exp(-|a|\lambda), \quad u_\lambda(x) = v_\lambda(x) = \sqrt{2/\pi}\sin \lambda x,$$

and, by the theorem, we have

$$\begin{aligned}\pi_{m-1}(x_m|a_{m-1}, \dots, a_1|x_1) &= \frac{2}{\pi}\int_0^\infty \exp\biggl\{-\lambda \sum_{j=1}^{m-1} |a_j|\biggr\} \sin \lambda y \sin \lambda x \, d\lambda \\ &= \pi_1\biggl(x_m \biggm| \sum_{j=1}^{m-1} |a_j| \biggm| x_1\biggr).\end{aligned} \tag{30}$$

Since the kernel (29) is nonnegative and

$$\int_0^\infty \pi_1(y|a_j|x)\, dy = \frac{2}{\pi}\arctan\frac{x}{|a_j|} \le 1, \tag{31}$$

then we can identify this kernel with the transition probability density of some Markov process by the formula

$$\pi_1(y|a_j|x) = \pi_{t_j \to t_{j+1}}(x \to y)$$

if we set $a_j = t_{j+1} - t_j$, $t_1 < t_2 < \dots < t_m$, because, by (30), the Chapman–Kolmogorov equation (20) holds.

This Markov process is of interest, because its trajectory issuing from an arbitrary finite point $x$ at the initial instant of time $s$ does not necessarily attain the interval $y \in [0, \infty)$ at any instant of time $t > s$ (by (31), the probability of this event is less than 1); note that the "shortage" of trajectories occurs due to inherent reasons, not because of boundaries. Only the trajectories issuing from the point at infinity (for $x = \infty$) at time $s$ remain in the interval $[0, \infty)$ at any instant of time $t > s$. If we use a metaphor identifying the Markov process generated by the pair $F(x)$, $\pi_{s \to t}(x \to y)$, where $F(x)$ is the distribution function of the initial values, with the transportation of the "water" $F(x)$ in the "vessel" $\pi_{s\to t}(x \to y)$, then, in the example under consideration, the Markov process is the transportation of the water in a sieve, while, in Example 1, the "water" is transported in an ordinary bucket without holes.

**Example 7** (with Fourier exponential transform)**.** In (1), suppose that

$$\Omega = (-\infty, \infty), \qquad \pi_1(y|a_j|x) = \frac{e^{-\alpha_j x}}{(e^{-y/\gamma_j} + e^{-x/\gamma_j})^{\nu_j}} \exp\{y(\alpha_j - \nu_j/\gamma_j)\}, \tag{32}$$

where $\operatorname{Re}\nu_j/\gamma_j > \operatorname{Re}\alpha_j > 0$, $a_j = (\alpha_j, \gamma_j, \nu_j)$. By [5, Vol. 1], the Fourier exponential transform of the kernel (32) is of the form

$$\int_{-\infty}^\infty \pi_1(y|a_j|x)e^{-ix\lambda}\, dx = \gamma_j e^{-i\lambda y} B(\gamma_j(\alpha_j + i\lambda), \nu_j - \gamma_j(\alpha_j + i\lambda)), \tag{33}$$

where $B(x, y)$ is the beta function. Inverting this transform, i.e., multiplying (33) by $\exp(i\lambda x)/2\pi$ and integrating over $\lambda$ in $(-\infty, \infty)$, we obtain the kernel $\pi_1(y|a_j|x)$ in the form

$$\pi_1(y|a_j|x) = \frac{\gamma_j}{2\pi}\int_{-\infty}^\infty e^{i\lambda(x-y)} B(\gamma_j(\alpha_j + i\lambda), \nu_j - \gamma_j(\alpha_j + i\lambda))\, d\lambda,$$

which coincides with representation (3) for $f(\lambda) = \lambda$, $\Lambda = (-\infty, \infty)$,

$$\psi_\lambda(a_j) = \gamma_j B\bigl(\gamma_j(\alpha_j + i\lambda), \nu_j - \gamma_j(\alpha_j + i\lambda)\bigr),$$

$$u_\lambda(x) = \frac{1}{\sqrt{2\pi}}\, e^{-i\lambda x}, \qquad v_\lambda(x) = \frac{1}{\sqrt{2\pi}}\, e^{i\lambda x}.$$

Therefore, by the theorem, the integral (1) can be written as the single integral (9).

**Example 8** (with the Hankel transform). In (1), suppose that $\Omega = (0, \infty)$,

$$\pi_1(y|a_j|x) = \frac{\sqrt{xy}}{r_1 r_2}\left(\frac{r_2 - r_1}{r_2 + r_1}\right)^{\nu},$$

where the $a_j$ are parameters,

$$\operatorname{Re} a_j > 0, \quad x > 0, \quad y > 0, \quad \operatorname{Re}\nu > -1,$$

while

$$r_1 = \sqrt{a_j^2 + (x - y)^2}\,, \qquad r_2 = \sqrt{a_j^2 + (x + y)^2}\,.$$

By [3, p. 686, no. 6.522(3)], we now have the representation

$$\pi_1(y|a|x) = \int_0^\infty K_0(a\lambda)\sqrt{y\lambda}\, J_\nu(y\lambda)\sqrt{x\lambda}\, J_\nu(x\lambda)\, d\lambda, \tag{34}$$

where $a = a_j$, $J_\nu(x)$ is the Bessel function of the first kind, and $K_0(x)$ is the Bessel function of the third kind (the MacDonald function). Setting

$$f(\lambda) = \lambda, \quad \Lambda = [0, \infty), \quad \psi_\lambda(a_j) = K_0(a_j\lambda), \quad u_\lambda(x) = v_\lambda(x) = \sqrt{x\lambda}\, J_\nu(x\lambda), \tag{35}$$

we see that the kernel (34) coincides with (3), so that, in view of the theorem, the integral (1) is of the form (9) in the notation (35).

**Example 9** (with the Kontorovich–Lebedev transform). In (1), suppose that

$$\Omega = (0, \infty), \qquad \pi_1(y|a_j|x) = y^{a_j - 1} x^{a_j} |x - y|^{-a_j} K_{a_j}(|x - y|), \tag{36}$$

where the $a_j$ are parameters, $0 < \operatorname{Re} a_j < 1/2$, and $K_a(x)$ is the MacDonald function. By [5, Vol. 2], we have

$$\int_0^\infty \pi_1(y|a|x)\frac{K_{i\gamma}(x)}{x}\, dx = \frac{1}{2^a y\sqrt{\pi}}\,\Gamma\left(\frac{1}{2} - a\right)\Gamma(a + i\gamma)\Gamma(a - i\gamma)K_{i\gamma}(y), \qquad a \equiv a_j.$$

Let us solve this integral equation for $\pi_1(y|a|x)$, using the pair of mutually inverse transformations [4], [5]

$$g(y) = \int_0^\infty f(x)K_{ix}(y)\, dx, \qquad f(x) = \frac{2}{\pi^2}\, x\sinh(\pi x)\int_0^\infty g(y)\frac{K_{i\xi}(y)}{y}\, dy.$$

This yields the equality

$$\pi_1(y|a|x) = \frac{2^{1-a}}{\pi^2\sqrt{\pi}}\,\Gamma\left(\frac{1}{2} - a\right)\int_0^\infty \Gamma(a + i\gamma)\Gamma(a - i\gamma)K_{i\gamma}(x)\frac{K_{i\gamma}(y)}{y}\,\gamma\sinh(\pi\gamma)\, d\gamma,$$

i.e., the kernel (36) can be expressed as (3) for

$$\lambda = \gamma, \quad f(\lambda) = \lambda, \quad a = a_j, \quad \Lambda = [0, \infty),$$

$$2^{a_j}\sqrt{\pi}\,\psi_\lambda(a_j) = \Gamma\left(\frac{1}{2} - a_j\right)\Gamma(a_j + i\gamma)\Gamma(a_j - i\gamma),$$

$$u_\lambda(x) = \frac{2\lambda}{\pi^2}\sinh(\pi\lambda)\frac{K_{i\lambda}(x)}{x}\,, \qquad v_\lambda(x) = K_{i\lambda}(x) \tag{37}$$

and, by the theorem, the integral (1) can be reduced to the single integral (9) in the notation (37).

Thus, the representation of the kernel in the form (3)–(5) allows us to reduce multiple integrals of the form (1) to single ones, using integral transform tables. The examples given above do not exhaust all the possibilities. So even a cursory glance at the tables dealing with Fourier cosine transforms [3] shows that there are about thirty kernels expressible as (3)–(5), while our examples involved only five kernels.

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