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An Asymptotic Series for the Weber–Schafheitlin Integral

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Abstract—We obtain an asymptotic power series for the Weber–Schafheitlin integral whose coefficients are distributions.

Key words: *Weber–Schafheitlin integral, Kontorovich–Lebedev transform, Gauss hypergeometric function, gamma-function, beta-function.*

In [1] we found the leading term of the asymptotics as $\rho \uparrow 1$ of the integral

$$I_c^\rho(\mu,\nu) = \int_0^\infty x^{-\rho} K_{i\mu}(x) K_{i\nu}(cx)\,dx, \tag{1}$$

in which $K_{i\lambda}(x)$ is the modified Bessel function of the third kind (the MacDonald function) [2 3], and $c > 0$, $\mu > 0$, $\nu > 0$. This integral belongs to the class of discontinuous Weber–Schafheitlin integrals [2], standing out among them as having the most complicated asymptotic behavior as $\rho \uparrow 1$. In the case $c = 1$, the asymptotic behavior of the integral (1) is given by

$$I_1^\rho(\mu,\nu) \to \frac{\pi^2}{(\mu+\nu)\sinh\dfrac{\pi(\mu+\nu)}{2}}\delta(\nu-\mu), \qquad \rho \uparrow 1 \tag{2}$$

(this property is used for the inversion of the Kontorovich–Lebedev transform [4]). The asymptotics (1) for $c \neq 1$ must be known in order to calculate nonlinear functionals from the trajectories of Gaussian Markov differentiable processes, for example, such functionals as the moments and the distribution function of the number of zeros of a process [5, 6]. In the case $c \neq 1$, the limiting (as $\rho \uparrow 1$) expression for $I_c^\rho(\mu,\nu)$ turns out to be more complicated than (2) and includes the distribution $\mathrm{Vp}(1/(\nu-\mu))$ in addition to the δ-function.

In the present paper, we obtain an asymptotic series for the integral (1) on the basis of the method used in [1], thus solving the problem of calculating this integral for ρ close to unity. For other values, $\rho < 1$, this integral is calculated using an exact formula (see [2, 3]), which involves the product of gamma-function, beta-function, and the Gauss hypergeometric function. Note that the use of this formula for finding an asymptotic series entails greater effort than the method proposed here.

Our result will involve the distributions $\delta(x)$, $\delta^{(n)}(x)$, and $\mathrm{Vp}\,x^{-n}$, where $n \geq 1$ is an integer. Using $\omega(x)$ to denote any one of these functions, we define them as linear continuous functionals $(\omega(x), \varphi(x))$ on the space $K(-b, a)$, $a > 0$, $b > 0$, of infinitely differentiable (on $(-b, a)$) functions

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$\varphi(x)$ assigning to $\varphi(x)$ the numbers on the right-hand sides of the following formulas:

$$(\delta(x),\varphi(x))=\varphi(0),\qquad (\delta^{(n)}(x),\varphi(x))=(-1)^n\frac{\varphi^{(n)}(0)}{n!},\tag{3}$$

$$\left(\mathrm{Vp}\,\frac{1}{x},\varphi(x)\right)=\mathrm{Vp}\int_{-b}^{a}\varphi(x)\,\frac{dx}{x},\tag{4}$$

$$\left(\mathrm{Vp}\,\frac{1}{x^{n+1}},\varphi(x)\right)=\mathrm{Vp}\int_{-b}^{a}\left[\varphi(x)-\sum_{k=0}^{n-1}\frac{\varphi^{(k)}(0)x^k}{k!}\right]\frac{dx}{x^{n+1}},\qquad n\ge 1.\tag{5}$$

By the symbol Vp in front of the integral we mean its principal value in the sense of Cauchy:

$$\mathrm{Vp}\int_{-b}^{a}f(x)\,dx=\lim_{\varepsilon\downarrow 0}\left(\int_{-b}^{-\varepsilon}f(x)\,dx+\int_{\varepsilon}^{a}f(x)\,dx\right).$$

For the distributions $\delta^{(n)}(x)$, the endpoints of the interval $[-b,a]$ for $a>0$, $b>0$ is of no importance, but for $\mathrm{Vp}\,x^{-n}$ their values are essential. For example,

$$\left(\mathrm{Vp}\,\frac{1}{x},1\right)=\mathrm{Vp}\int_{-b}^{a}\frac{dx}{x}=\ln\frac{a}{b}.\tag{6}$$

We shall say that a *function* $f(\alpha,x)$ *is expandable in a generalized asymptotic series in the space* $K(-b,a)$ and write

$$f(\alpha,x)\sim\sum_{n=0}^{\infty}A_n(x)\omega_n(\alpha),\qquad \alpha\downarrow 0$$

if $\omega_{n+1}(\alpha)=o(\omega_n(\alpha))$ and for any function $\varphi(x)\in K(-b,a)$ the series

$$(f(\alpha,x),\varphi(x))\sim\sum_{n=0}^{\infty}(A_n(x),\varphi(x))\omega_n(\alpha),\qquad \alpha\downarrow 0,$$

is an asymptotic series (in the ordinary sense) for the functional $(f(\alpha,x),\varphi(x))$.

Theorem. *For* $c>0$, $\mu>0$, $\nu>0$, *the function* $f(\alpha,\beta)\equiv I_c^\rho(\mu,\nu)$ *can be expanded in the following generalized asymptotic series in the space* $K(-b,a)$:

$$I_c^\rho(\mu,\nu)\sim\frac{\pi}{4\gamma\sinh\pi\gamma}\sum_{n=0}^{\infty}A_n(\beta|\gamma,c)\frac{\alpha^n}{n!},\qquad \alpha=\frac{1-\rho}{2}\downarrow 0,$$

where $2\beta=\nu-\mu$, $2\gamma=\nu+\mu$, $\beta\in[-b,a]$,

$$A_n(\beta|\gamma,c)=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}a_k(\gamma)B_{n-k}(\beta|\gamma,c),\tag{7}$$

$$a_n(\gamma)=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(\ln 4)^{n-k}b_k,\tag{8}$$

$$b_0=1,\quad b_{n+1}\equiv b_{n+1}(\gamma)=v_n+\sum_{k_1=1}^{n}\frac{n!}{(n-k_1)!\,k_1!}(1-\delta_{n0})v_{n-k_1}v_{k_1-1}$$

$$+\sum_{j=2}^{n}\sum_{k_1=j}^{n}\sum_{k_2=j-1}^{k_1-1}\cdots\sum_{k_j=1}^{k_{j-1}-1}\frac{n!(1-\delta_{n0})}{(n-k_1)!\,k_j!}v_{n-k_1}v_{k_j-1}\prod_{r=1}^{j-1}\frac{v_{k_r-k_{r+1}-1}(1-\delta_{nr})}{(k_r-k_{r+1}-1)!k_r},\tag{9}$$

$$v_n \equiv v_n(\gamma) = \psi^{(n)}(i\gamma) + \psi^{(n)}(-i\gamma), \qquad \psi^{(n)}(x) = \frac{d^n}{dx^n} \ln \Gamma(x),$$

$$B_n(\beta|\gamma, c) = K_n(\beta, \gamma;\, c) + \pi\delta^{(n)}(\beta) \cos\left(\frac{n\pi}{2} - \gamma \ln c\right) + n![\operatorname{Vp} \beta^{-n-1} - d_n(\beta)] \sin\left(\frac{n\pi}{2} + \gamma \ln c\right), \tag{10}$$

$$d_0(\beta) = 0, \quad d_n(\beta) = \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(n-k)!k} \delta^{n-k}(\beta)[a^{-k} - (-b)^{-k}], \tag{11}$$

$$K_n(\beta, \gamma;\, c) = \int_0^1 \left[\cos\left(\beta \ln x - \gamma \ln \frac{1+cx}{x+c}\right) \ln^n \frac{x}{(x+c)(1+cx)} - \cos(\beta \ln x + \gamma \ln c) \ln^n \frac{x}{c}\right] \frac{dx}{x}, \tag{12}$$

$\Gamma(x)$ is the gamma-function, $\psi^{(n)}(x)$ is the polygamma-function, and δ_{ij} is the Kronecker delta.

Remark. The coefficients of (9) $b_n \equiv b_n(\gamma)$ are defined by the relation (see Lemma 4 below)

$$\frac{d^n}{d\alpha^n} \Gamma(\alpha + i\gamma)\, \Gamma(\alpha - i\gamma)\Big|_{\alpha=0} = \Gamma(i\gamma)\, \Gamma(-i\gamma)\, b_n(\gamma).$$

The first four coefficients are

$$b_1 = v_0, \quad b_2 = v_1 + v_0^2, \quad b_3 = v_2 + 3v_1v_0 + v_0^3, \quad b_4 = v_3 + 4v_2v_0 + 6v_1v_0^2 + 3v_1^2 + v_0^4,$$

where the $v_k(\gamma)$ are the same as in the statement of the theorem.

The proof of the theorem is based on Lemmas 1–4, which can be of use outside the scope of the problem under consideration.

Lemma 1. *For $\varphi(x) \in K(-b, a)$, the following formula is valid:*

$$\int_{-b}^{a} c_n(x) \frac{dx}{x^{n+1}} = \left(\operatorname{Vp} \frac{1}{x^{n+1}}, \varphi(x)\right) - \frac{\varphi^{(n)}(0)(\operatorname{Vp} 1/x, 1)}{n!}, \tag{13}$$

in which

$$c_n(x) = \varphi(x) - \sum_{k=0}^{n} \frac{\varphi^{(k)}(0)x^k}{k!}. \tag{14}$$

Indeed, by the definition of $\operatorname{Vp} x^{-n-1}$ (see (5)) we have

$$\left(\operatorname{Vp} \frac{1}{x^{n+1}}, \varphi(x)\right) = \operatorname{Vp} \int_{-b}^{a} \left[c_n(x) + \frac{\varphi^{(n)}(0)x^n}{n!}\right] \frac{dx}{x^{n+1}}. \tag{15}$$

Since $c_n(x) = O(x^{n+1})$ for small x, it follows that

$$\operatorname{Vp} \int_{-b}^{a} c_n(x) x^{-n-1}\, dx = \int_{-b}^{a} c_n(x) x^{-n-1}\, dx, \tag{16}$$

and we readily obtain (13) from (15) and (16), since the integral of the second summand in (15) is equal to $\varphi^{(n)}(0)(\operatorname{Vp} 1/x, 1)/n!$.

Lemma 2. *We have the following generalized asymptotic series in the space* $K(-b,a)$:

$$\frac{1}{x \mp i\alpha} \sim \sum_{n=0}^{\infty} (\pm i\alpha)^n [(x \mp i0)^{-n-1} - d_n(x)], \qquad \alpha \downarrow 0, \tag{17}$$

where the $d_n(x)$ *are defined by formula* (11) (*for* $\beta = x$) *and*

$$(x \mp i0)^{-n-1} = \operatorname{Vp} x^{-n-1} \pm \frac{(-1)^n i\pi\delta^{(n)}(x)}{n!}. \tag{18}$$

Proof. Consider the linear functional $(1/(x-i\alpha), \varphi(x))$ for $\varphi(x) \in K(-b,a)$. Obviously,

$$\left(\frac{1}{x-i\alpha}, \varphi(x)\right) = \varphi(i\alpha)\left(\frac{1}{x-i\alpha}, 1\right) + \left(1, \frac{\varphi(x)-\varphi(i\alpha)}{x-i\alpha}\right). \tag{19}$$

For $\alpha \neq 0$, we have

$$\left(\frac{1}{x-i\alpha}, 1\right) = \int_{-b}^{a} \frac{dx}{x-i\alpha} = \frac{1}{2}\ln\frac{a^2+\alpha^2}{b^2+\alpha^2} + i\left[\arctan\frac{a}{\alpha} + \arctan\frac{b}{\alpha}\right]$$

and this expression can be expanded in the asymptotic series

$$\ln\frac{a}{b} + i\pi - \sum_{k=1}^{\infty} (i\alpha)^k \frac{a^{-k} - (-b)^{-k}}{k}, \qquad \alpha \downarrow 0.$$

In a similar way (Taylor series), we obtain

$$\frac{\varphi(x)-\varphi(i\alpha)}{x-i\alpha} = \sum_{n=0}^{\infty} (i\alpha)^n c_n(x) x^{-n-1}, \tag{20}$$

where the coefficients $c_n(x)$ are the same as in Lemma 1 (see (14)). Using this lemma and the linearity of the functional, for the second summand in (19) we obtain

$$\left(1, \frac{\varphi(x)-\varphi(i\alpha)}{x-i\alpha}\right) = \sum_{n=0}^{\infty} (i\alpha)^n \left(\operatorname{Vp}\frac{1}{x^{n+1}}, \varphi(x)\right) - \varphi(i\alpha)\left(\operatorname{Vp}\frac{1}{x}, 1\right), \tag{21}$$

so that, by (20), (21), and (6), from (19) we obtain

$$\left(\frac{1}{x-i\alpha}, \varphi(x)\right) = \varphi(i\alpha)\left\{i\pi - \sum_{k=1}^{\infty}\frac{(i\alpha)^k}{k}[a^{-k} - (-b)^{-k}] + \sum_{n=0}^{\infty}(i\alpha)^n\left(\operatorname{Vp}\frac{1}{x^{n+1}}, \varphi(x)\right)\right\}. \tag{22}$$

Expressing $\varphi(i\alpha)$ as a Taylor series in powers of α and multiplying the asymptotic series in the first summand in (22), we see that

$$\left(\frac{1}{x-i\alpha}, \varphi(x)\right) = \sum_{n=0}^{\infty}(i\alpha)^n\left\{\frac{\varphi^{(n)}(0)}{n!}i\pi + \left(\operatorname{Vp}\frac{1}{x^{n+1}}, \varphi(x)\right) - \sum_{k=1}^{n}\frac{\varphi^{(n-k)}(0)}{(n-k)!k}[a^{-k} - (-b)^{-k}]\right\}.$$

From this expression we obtain (17) if we recall (3)–(5), (18), and (11). In the same way, we can prove the formula for $1/(x+i\alpha)$. □

Note that the finiteness of the length of the interval $(-b,a)$ in the definition of the space $K(-b,a)$ of main functions gives rise to the additional summands $d_n(x)$ in (17), as compared to formulas generalizing the Sokhotskii formulas [7].

Lemma 3. *The following generalized asymptotic series in the space $K(-b,a)$ are valid*:

$$\frac{\alpha}{x^2+\alpha^2} \sim \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \left[\pi\delta^{(n)}(x)\cos\frac{n\pi}{2} + n!\left(\operatorname{Vp}\frac{1}{x^{n+1}} - d_n(x)\right)\sin\frac{n\pi}{2}\right], \qquad \alpha \downarrow 0, \tag{23}$$

$$\frac{x}{x^2+\alpha^2} \sim \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \left[\pi\delta^{(n)}(x)\sin\frac{n\pi}{2} + n!\left(\operatorname{Vp}\frac{1}{x^{n+1}} - d_n(x)\right)\cos\frac{n\pi}{2}\right], \qquad \alpha \downarrow 0, \tag{24}$$

where the $d_n(x)$ are defined by relations (11) (*for $x=\beta$*).

Formulas (23), (24) are readily obtained from the representation

$$\frac{2i\alpha}{x^2+\alpha^2} = \frac{1}{x+i\alpha} - \frac{1}{x-i\alpha}, \qquad \frac{2x}{x+i\alpha} = \frac{1}{x+i\alpha} + \frac{1}{x-i\alpha}$$

and the previous Lemma 2 (see (17)–(19)).

Lemma 4. *If $\gamma > 0$, then we have the convergent series*

$$4^{\alpha-1}\Gamma(\alpha+i\gamma)\,\Gamma(\alpha-i\gamma) = \frac{\pi}{4\gamma\sinh\pi\gamma}\sum_{n=0}^{\infty} a_n(\gamma)\frac{\alpha^n}{n!}, \tag{25}$$

in which the $a_n(\gamma)$ are defined by relations (8) (9).

This assertion follows from the analyticity of the functions on the left-hand side of (25) for $\gamma > 0$. Multiplying the Taylor series for $4^{\alpha-1}$ by $\Gamma(\alpha+i\gamma)\,\Gamma(\alpha-\gamma)$, we obtain (25), since by the Leibniz rule for the differentiation of a product we have

$$\frac{d^n}{d\alpha^n}\Gamma(\alpha+i\gamma)\,\Gamma(\alpha-i\gamma)\Big|_{\alpha=0} = \Gamma(i\gamma)\,\Gamma(-i\gamma)\,b_n(\gamma),$$

where $\Gamma(i\gamma)\,\Gamma(-i\gamma) = \pi/(\gamma\sinh\pi\gamma)$ and the coefficients $b_n(\gamma)$ are the same as in (9).

Proof of the theorem. In [1], it was shown that

$$I_c^{\rho}(\mu,\nu) = 4^{\alpha-1}\Gamma(\alpha+i\gamma)\,\Gamma(\alpha-i\gamma)\,F_c^{\alpha}(\beta,\gamma),$$

where

$$F_c^{\alpha}(\beta,\gamma) = \int_{-\infty}^{\infty} \frac{e^{2it\beta}\psi_c^{\gamma}(e^t)\,dt}{(1+2c\cdot\cosh 2t+c^2)^{\alpha}}, \qquad \psi_c^{\gamma}(x) = \left(\frac{x^2+c}{1+cx^2}\right)^{i\gamma}.$$

In view of Lemma 4, it only remains to find an asymptotic series for the integral $F_c^{\alpha}(\beta,\gamma)$. By the change of variable $\exp(-2t) = x$, we reduce the integral to the form

$$2F_c^{\alpha}(be,\gamma) = \int_0^1 x^{\alpha-1}[f_\alpha(x|\beta,\gamma\,;\,c) + f_\alpha(x|-\beta,-\gamma\,;\,c)]\,dx, \tag{26}$$

where

$$f_\alpha(x|\beta,\gamma\,;\,c) = x^{-i\beta}(x+c)^{-i\gamma-\alpha}(1+cx)^{i\gamma-\alpha}.$$

In (26), adding to the first and second summands in brackets and subtracting from them $x^{-i\beta}c^{-i\gamma}$ and $x^{i\beta}c^{i\gamma}$, respectively, and taking the integrals of $x^{\alpha-i\beta-1}$ and $x^{\alpha+i\beta-1}$, we obtain

$$2F_c^{\alpha}(\beta,\gamma) = \frac{c^{-i\gamma}}{\alpha-i\beta} + \frac{c^{i\gamma}}{\alpha+i\beta} + J_c^{\alpha}(\beta,\gamma) + J_c^{\alpha}(-\beta,-\gamma), \tag{27}$$

$$J_c^{\alpha}(\beta,\gamma) = \int_0^1 \left[\left(\frac{1+cx}{x+c}\right)^{i\gamma}\exp\left(\alpha\ln\frac{x}{(x+c)(1+cx)}\right) - c^{-i\gamma}\exp\left(\alpha\ln\frac{a}{b}\right)\right]\frac{dx}{x^{i\beta+1}}. \tag{28}$$

In (28), let us replace the exponentials by their Taylor series in powers of α. We obtain the asymptotic series

$$J_c^\alpha(\beta,\gamma) \sim \sum_{n=0}^{\infty} Q_n(\beta,\gamma\,;\,c)\frac{\alpha^n}{n!}, \qquad \alpha \downarrow 0, \tag{29}$$

in which

$$Q_n(\beta,\gamma\,;\,c) = \int_0^1 x^{-i\beta-1}\left[\left(\frac{1+cx}{x+c}\right)^{i\gamma} \ln^n \frac{x}{(x+c)(1+cx)} - c^{-i\gamma}\frac{\ln^n x}{c}\right] dx. \tag{30}$$

The integrals (30) converge for any integer n and any real β and $\gamma > 0$. Indeed, the integrand can have a singularity only at $x = 0$, but in the neighborhood of zero this function for $\gamma > 0$ is of order $x^{-i\beta}\ln^n x$ and this singularity is integrable for a real β.

It is readily verified that the series (29) is asymptotic if we replace the Taylor series for the exponentials by partial sums whose remainder, after integration over $(0,1)$, yields a finite value multiplied by the order of the first discarded term.

Since the first two and the last two summands in (27) are complex conjugate values, from (27), by (29), (30), we obtain

$$F_c^\alpha(\beta,\gamma) \sim \frac{\alpha}{\alpha^2+\beta^2}\cos(\gamma\ln c) + \frac{\beta}{\alpha^2+\beta^2}\sin(\gamma\ln c) + \sum_{n=0}^{\infty} K_n(\beta,\gamma\,;\,c)\frac{\alpha^n}{n!}, \tag{31}$$

where the $K_n(\beta,\gamma\,;\,c)$ are defined by relation (12).

The asymptotic series as $\alpha \downarrow 0$ for the first two summands in (31) were constructed in Lemma 3. Using (23)–(25), from (31) we obtain the generalized asymptotic series

$$F_c^\alpha(\beta,\gamma) \sim \sum_{n=0}^{\infty} B_n(\beta|\gamma\,;\,c)\frac{\alpha^n}{n!} \tag{32}$$

on the functions $\varphi(\beta) \in K(-b,a)$.

Multiplying the asymptotic series (32) and (25) (Lemma 4) and collecting the coefficients of α^n, we obtain the assertion of the theorem. This operation is legitimate, since the series in powers of α are multiplied and since only one of them is a generalized asymptotic series. □

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