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Two-component flow round a solid sphere

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The processes of mass and heat transfer are important in many practical situations. In order to solve these problems we have to know first of all the velocity distribution of appropriate flows. On the other hand the named processes take often place in the immediate neighbourhood of immersed particles like solid particles, drops, bubbles and so on. Therefore the problem of two-component flow round a solid sphere is of great importance.

Starting from the model idea for multicomponent flows as presented by L. D. Landau [1] and Ch. A. Rachmatulin [2] in this paper at first the solution of the hydrodynamic problem of this motion is given. In this case the formulas for velocities and pressures of the components and of the mixture are derived and a generalization of Stokes formula for the force on a small sphere is found.

In this connection it is also possible to solve the inverse problem, namely to find the mixture viscosity μ_G and the interaction coefficient K between the components.

In the second part of the paper the problem of convective mass transport in a two-component flow round a solid sphere is investigated and a formula for the total diffusion stream on a small sphere is given.

1. The formulation of the problem

We consider a sphere with radius a in a stationary laminar incompressible two-component flow. We assume that the real and reduced densities of both the components are constant, namely $\varrho_i^* = \text{const}$ and $\varrho_i = \text{const}$. Therefore the porosities $\varkappa_i = \dfrac{\varrho_i}{\varrho_i^*}$ are constant, too. The interaction coefficient K was also assumed to be constant.

Now let us consider a two-component flow with small Reynolds numbers $\mathrm{Re}_i \ll 1$, i.e. the sphere radius a is assumed to be sufficiently small. Then in the equations of motion the inertia terms can be neglected in comparison with the friction and interaction terms. If in Rachmatulin's equations we eliminate the inertia terms and we assume that external forces do not exist the equation system will be given by

$$\operatorname{grad} p = \mu_i \, \Delta \boldsymbol{v}_i + \frac{K}{\varkappa_i} \sum_{j=1}^{2} (\boldsymbol{v}_j - \boldsymbol{v}_i), \quad \operatorname{div} \boldsymbol{v}_i = 0, \quad i = 1, 2, \tag{1.1}$$

where μ_i, $\boldsymbol{v}$, p are adequate the viscosities, velocities and pressure of both components.

For the complete formulation of the problem we need, besides the equations (1.1), the boundary conditions on the sphere surface and far from the sphere.

For the further investigations we choose the coordinate system so that the coordinate origin is placed at the centre of the sphere and the motion takes place parallel to the x-axis. Because of the form of the regarded body it is suitable to carry out the investigations in spherical coordinates (r, θ, φ). Therefore we have an axial symmetric flow for which

$$\frac{\partial \boldsymbol{v}_i}{\partial \varphi} = 0, \quad \boldsymbol{i}_\varphi \cdot \boldsymbol{v}_i = v_{i\varphi} = 0 \tag{1.2}$$

is valid and for which the x-axis is the axis of symmetry.

As is shown in [3] for such flows the stream functions ψ_i and ψ exist for the single components and for the mixture. With the help of these stream functions the different parameters of the two-component flow can be determined.

Therefore the problem is to solve the equations for the stream functions ψ_i derived from (1.1) considering the appropriate boundary conditions.

2. The velocities of the components and the mixture of the two-component medium

By the definition for the stream function the velocity components are given by

$$v_{ir} = -\frac{1}{r^2 \sin\theta}\frac{\partial \psi_i}{\partial \theta}, \quad v_{i\theta} = \frac{1}{r \sin\theta}\frac{\partial \psi_i}{\partial r}. \tag{2.1}$$

In [3] is shown that the equations of motion (1.1) can be decoupled and regarding the relations (2.1) the equations for the determination of the stream functions become the following:

$$E^4 \chi_1 = 0, \quad E^2[E^2 \chi_2 - \gamma^2 \chi_2] = 0. \tag{2.2}$$

The operator E^2 in (2.2) is given by

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\eta^2}{r^2}\frac{\partial^2}{\partial \eta^2}, \quad \eta = \cos\theta. \tag{2.3}$$

There are

$$\chi_1 = \varkappa_1 \psi_1 + \varkappa_2 \mu^* \psi_2, \quad \chi_2 = \psi_1 - \psi_2, \tag{2.4}$$

and

$$\mu^* = \frac{\mu_2}{\mu_1}, \quad \gamma^2 = K_1 + K_2, \quad K_i = \frac{K}{\mu_i \varkappa_i}. \tag{2.5}$$

If we now determine the functions χ_i from (2.2), we shall, according to (2.3), find the stream functions ψ_i, namely

$$\psi_1 = \frac{\chi_1 + \varkappa_2 \mu^* \chi_2}{\varkappa_1 + \varkappa_2 \mu^*}, \quad \psi_2 = \frac{\chi_1 - \varkappa_1 \chi_2}{\varkappa_1 + \varkappa_2 \mu^*}. \tag{2.6}$$

With the use of the formula $\psi = \varkappa_1 \psi_1 + \varkappa_2 \psi_2$ we get the stream function of the mixture

$$\psi = \frac{1}{\varkappa_1 + \varkappa_2 \mu^*} \left[\chi_1 + \varkappa_1 \varkappa_2 (\mu^* - 1) \chi_2 \right]. \tag{2.7}$$

Considering the relations (2.6) we can compute the velocities of the components according to (2.1). Then the velocities of the mixture are given by the formulas

$$V_{q_k} = \varkappa_1 v_{1q_k} + \varkappa_2 v_{2q_k}, \quad k = 1, 2, \quad q_1 = r, \quad q_2 = \theta. \tag{2.8}$$

As in [3] is shown the general solution for the motion of a spherical particle in a two-component medium can be found by solving the equations (2.2). We then have

$$\chi_1 = \frac{1}{2} \sin^2 \theta \, (A_1 r^2 + B_1 r^{-1} + C_1 r^4 + D_1 r),$$
$$\chi_2 = \frac{1}{2} \sin^2 \theta \left[A_2 \left(\gamma - \frac{1}{r}\right) e^{\gamma r} + B_2 \left(\gamma + \frac{1}{r}\right) e^{-\gamma r} - \frac{1}{\gamma^2} (C_2 r^2 + D_2 r^{-1}) \right]. \tag{2.9}$$

For the fixation of the boundary conditions we now consider the following problem: A motionless solid sphere is in the flow of a two-component medium the components of which have the velocities $\vec{U}_i = -\vec{i}_x U_i$ far from the sphere.

The boundary conditions on the sphere surface are the following

$$\psi_i|_{r=a} = 0, \quad \left.\frac{\partial \psi_i}{\partial r}\right|_{r=a} = 0. \tag{2.10}$$

Because the flow field far from the sphere is unperturbed the following relations must be satisfied there:

$$\psi_i \to \frac{1}{2} r^2 U_i \sin^2 \theta \quad \text{for} \quad r \to \infty. \tag{2.11}$$

In view of (2.4), (2.10) and (2.11) we obtain the boundary conditions for χ_1 and χ_2

$$\chi_1|_{r=a} = 0, \quad \left.\frac{\partial \chi_1}{\partial r}\right|_{r=a} = 0, \quad \chi_1 \to \frac{1}{2} r^2 U^* \sin^2 \theta \quad \text{for} \quad r \to \infty, \tag{2.12}$$

and

$$\chi_2|_{r=a} = 0, \quad \left.\frac{\partial \chi_2}{\partial r}\right|_{r=a} = 0, \quad \chi_2 \to \frac{1}{2} r^2 \bar{U} \sin^2 \theta \quad \text{for} \quad r \to \infty, \tag{2.13}$$

where

$$U^* = \varkappa_1 U_1 + \varkappa_2 \mu^* U_2, \quad \bar{U} = U_1 - U_2. \tag{2.14}$$

With the use of these boundary conditions we now determine χ_1 and χ_2 from (2.9). As according to (2.12) $\frac{\chi_1}{r_2} \to \frac{1}{2} U^* \sin^2 \theta$ for $r \to \infty$ we obtain

$$C_1 = 0, \quad A_1 = U^*$$

immediately from (2.9), so that χ_1 is given by

$$\chi_1 = \frac{1}{2} \sin^2 \theta \, (U^* r^2 + B_1 r^{-1} + D_1 r). \tag{2.15}$$

The conditions (2.12) for $r = a$ supply us with equations for the determination of B_1 and D_1, namely

$$\frac{B_1}{a} + D_1 a = -a^2 U^*, \quad -\frac{B_1}{a^2} + D_1 = -2aU^*,$$

from which we find

$$B_1 = \frac{1}{2} a^3 U^*, \quad D_1 = -\frac{3}{2} aU^*.$$

If we put the obtained coefficients into (2.15), we shall get to the relation

$$\chi_1 = \frac{1}{2} r^2 U^* \sin^2\theta \left[\frac{1}{2}\left(\frac{a}{r}\right)^3 - \frac{3}{2}\left(\frac{a}{r}\right) + 1\right]. \tag{2.16}$$

When we consider the boundary conditions (2.13) for χ_2 we see that $\frac{\chi_2}{r_2} \to \frac{1}{2}\,\overline{U}\sin^2\theta$ for $r \to \infty$. Because of (2.9) there are then

$$A_2 = 0, \quad C_2 = -\gamma^2 \overline{U},$$

so that we can write for χ_2

$$\chi_2 = \frac{1}{2}\sin^2\theta\left[B_2\left(\gamma + \frac{1}{r}\right) e^{-\gamma r} + \overline{U} r^2 - D_2 \frac{1}{\gamma^2 r}\right]. \tag{2.17}$$

Then the conditions (2.13) for $r = a$ supply us with the following equations for the determination of B_2 and D_2

$$B_2\left(\gamma + \frac{1}{a}\right) e^{-\gamma a} - \frac{1}{\gamma^2}\frac{D_2}{a} = -a_2 \overline{U},$$

$$-B_2\left[\frac{1}{a^2} + \gamma\left(\gamma + \frac{1}{a}\right)\right] e^{-\gamma a} + \frac{1}{\gamma^2}\frac{D_2}{a^2} = -2a\overline{U}.$$

After solving this system we find

$$B_2 = \frac{3a\overline{U}}{\gamma^2} e^{\gamma a}, \quad D_2 = a\overline{U}[\gamma^2 a^2 + 3(\gamma a + 1)].$$

If we put these coefficients into (2.17), we shall obtain, after some transformations, the formula

$$\chi_2 = \frac{1}{2} r^2 \overline{U} \sin^2\theta \left\{1 - \left(\frac{a}{r}\right)^3 \left[1 + \frac{3}{\gamma^2 a^2}\left((\gamma a + 1) - (\gamma r + 1) e^{-\gamma(r-a)}\right)\right]\right\}. \tag{2.18}$$

The χ_1 and χ_2 having been found, the stream functions ψ_1 and ψ_2 can be determined from (2.6) and the velocities of the components can be found from (2.1). We then have

$$\left.\begin{aligned} v_{1r} = -\frac{\cos\theta}{\varkappa_1 + \varkappa_2\mu^*} &\left\{U^*\left[\frac{1}{2}\left(\frac{a}{r}\right)^3 - \frac{3}{2}\left(\frac{a}{r}\right) + 1\right]\right. \\ &\left. + \varkappa_2\mu^*\overline{U}\left[1 - \left(\frac{a}{r}\right)^3\left[1 + \frac{3}{\gamma^2 a^2}\left((\gamma a + 1) - (\gamma r + 1) e^{-\gamma(r-a)}\right)\right]\right]\right\}, \end{aligned}\right\} \tag{2.19}$$

$$\left.\begin{aligned}
v_{2r} &= -\frac{\cos\theta}{\varkappa_1+\varkappa_2\mu^*}\left\{U^*\left[\frac{1}{2}\left(\frac{a}{r}\right)^3-\frac{3}{2}\left(\frac{a}{r}\right)+1\right]\right.\\
&\qquad\left.-\varkappa_1\overline{U}\left[1-\left(\frac{a}{r}\right)^3\left[1+\frac{3}{\gamma^2a^2}\left((\gamma a+1)-(\gamma r+1)\,\mathrm{e}^{-\gamma(r-a)}\right)\right]\right]\right\},\\
v_{1\theta} &= \frac{\sin\theta}{2(\varkappa_1+\varkappa_2\mu^*)}\left\{U^*\left[-\frac{1}{2}\left(\frac{a}{r}\right)^3-\frac{3}{2}\left(\frac{a}{r}\right)+2\right]\right.\\
&\qquad\left.+\varkappa_2\mu^*\overline{U}\left[2+\left(\frac{a}{r}\right)^3\left[1+\frac{3}{\gamma^2a^2}\left((\gamma a+1)-(\gamma^2r^2+\gamma r+1)\,\mathrm{e}^{-\gamma(r-a)}\right)\right]\right]\right\},\\
v_{2\theta} &= \frac{\sin\theta}{2(\varkappa_1+\varkappa_2\mu^*)}\left\{U^*\left[-\frac{1}{2}\left(\frac{a}{r}\right)^3-\frac{3}{2}\left(\frac{a}{r}\right)+2\right]\right.\\
&\qquad\left.-\varkappa_1\overline{U}\left[2+\left(\frac{a}{r}\right)^3\left[1+\frac{3}{\gamma^2a^2}\left((\gamma a+1)-(\gamma^2r^2+\gamma r+1)\,\mathrm{e}^{-\gamma(r-a)}\right)\right]\right]\right\}.
\end{aligned}\right\}\qquad(2.19)$$

In view of these expressions we can calculate the velocities of the mixture from (2.8), namely

$$\left.\begin{aligned}
v_r &= -\frac{\cos\theta}{\varkappa_1+\varkappa_2\mu^*}\left\{U^*\left[\frac{1}{2}\left(\frac{a}{r}\right)^3-\frac{3}{2}\left(\frac{a}{r}\right)+1\right]\right.\\
&\qquad\left.+\varkappa_1\varkappa_2(\mu^*-1)\,\overline{U}\left[1-\left(\frac{a}{r}\right)^3\left[1+\frac{3}{\gamma^2a^2}\left((\gamma a+1)-(\gamma r+1)\,\mathrm{e}^{-\gamma(r-a)}\right)\right]\right]\right\},\\
v_\theta &= \frac{\sin\theta}{2(\varkappa_1+\varkappa_2\mu^*)}\left\{U^*\left[-\frac{1}{2}\left(\frac{a}{r}\right)^3-\frac{3}{2}\left(\frac{a}{r}\right)+2\right]\right.\\
&\qquad+\varkappa_1\varkappa_2(\mu^*-1)\,\overline{U}\left[2+\left(\frac{a}{r}\right)^3\right.\\
&\qquad\left.\left.\times\left[1+\frac{3}{\gamma^2a^2}\left((\gamma a+1)-(\gamma^2r^2+\gamma r+1)\,\mathrm{e}^{-\gamma(r-a)}\right)\right]\right]\right\}.
\end{aligned}\right\}\qquad(2.20)$$

If we put $U_1=U_2$, $\mu_1=\mu_2$ and $\varrho_1^*=\varrho_2^*$ into (2.19) and (2.20), i.e. if we have a quasi-homogeneous flow, the velocities shall be indepedent on the interaction between the components. In this case we get formulas which were obtained for a homogeneous medium [4].

3. The generalization of Stokes formula for the drag force of the small sphere and the determination of μ_G, K and the pressure of the two-component medium

For the case of an axial symmetric flow J. Happel and H. Brenner [4] have shown that the force acting on a small particle can be computed by

$$F_1=\pi\mu_G\int\xi^3\,\frac{\partial}{\partial n}\left(\frac{E^2\psi}{\xi^2}\right)ds. \qquad(3.1)$$

For a small sphere this expression becomes

$$F_1 = \pi\mu_G \int_0^\pi a^3 \sin^3\theta \frac{\partial}{\partial r}\left(\frac{E^2\psi}{r^2\sin^2\theta}\right)\Bigg|_{r=a} a\, d\theta, \tag{3.2}$$

where μ_G is the mixture viscosity. On the other hand in [3] is explained that the force acting on a small particle for axial symmetric flows of a two-component medium can be found by the formula

$$F_2 = \pi\mu_1 \int \xi^3 \frac{\partial}{\partial n}\left(\frac{E^2\chi_1}{\xi^2}\right) ds \tag{3.3}$$

which for a small sphere is expressed by

$$F_2 = \pi\mu_1 \int_0^\pi a^3 \sin^3\theta \frac{\partial}{\partial r}\left(\frac{E^2\chi_1}{r^2\sin^2\theta}\right)\Bigg|_{r=a} a\, d\theta. \tag{3.4}$$

If we now consider our problem as a flow of a homogeneous medium round the sphere, we shall obtain, according to Stokes, the following formula of force on the sphere:

$$F_3 = -6\pi a\mu_G U_G, \tag{3.5}$$

where $U_G = \varkappa_1 U_1 + \varkappa_2 U_2$.

In view of (2.7), (2.16) and (2.18) we see that the expression for F_2 includes only the parameters of both the components, but F_1 additionally depends on μ_G and K and F_3 on μ_G. The mixture viscosity μ_G and the interaction coefficient K are not known. Since all three formulas (3.2), (3.4) and (3.5) represent the force on the sphere for the flow of the two-component medium, they have to be $F_1 = F_2 = F_3$. When we set $F_1 = F_3$ then we get the equation for the determination of K. The relation $F_2 = F_3$ gives us an equation for the compution of μ_G. If we put μ_G and K obtained in this way into (3.2), the relation $F_1 = F_2$ must be satisfied.

Now we compute the force F_2 according to (3.4). This force represents the generalization of the Stokes formula of the force on the sphere for the case of two-component flow.

Applying the operator E^2 according to (2.3) to (2.16) we obtain

$$E^2\chi_1 = \frac{3}{2} aU^* \frac{\sin^2\theta}{r}. \tag{3.6}$$

When we set this expression into (3.4) and we then integrate, the drag force of the sphere is given by

$$F = -6\pi a\mu_1(\varkappa_1 U_1 + \varkappa_2\mu^* U_2), \tag{3.7}$$

where it is assumed $F_2 = F$.

Now we compute F_1 from (3.2). According to (2.7) there is

$$\psi = \alpha\chi_1 + \beta\chi_2, \tag{3.8}$$

where

$$\alpha = \frac{1}{\varkappa_1 + \varkappa_2\mu^*}, \quad \beta = \frac{\varkappa_1\varkappa_2(\mu^* - 1)}{\varkappa_1 + \varkappa_2\mu^*}. \tag{3.9}$$

Applying the operator E^2 to (3.8) we obtain

$$E^2\psi = \alpha E^2\chi_1 + \beta E^2\chi_2. \tag{3.10}$$

With the use of (3.6) we can determine $E^2\chi_1$.

Applying (2.3) to (2.18) we find

$$E^2\chi_2 = \frac{3}{2}\, a\bar{U} \sin^2\theta \left(\gamma + \frac{1}{r}\right) e^{-\gamma(r-a)}. \tag{3.11}$$

The formulas (3.6) and (3.11) put into (3.10) give us

$$E^2\psi = \frac{3}{2}\, a \sin^2\theta \left[\alpha \frac{U^*}{r} + \beta\bar{U}\left(\gamma + \frac{1}{r}\right) e^{-\gamma(r-a)}\right]. \tag{3.12}$$

In view of this expression in (3.2) we obtain, after integration, F_1 as follows:

$$F_1 = -\frac{6\pi a\mu_G}{\varkappa_1 + \varkappa_2\mu^*}\Big[(\varkappa_1 U_1 + \varkappa_2\mu^* U_2) + \frac{1}{3}\varkappa_1\varkappa_2(\mu^* - 1) \times (U_1 - U_2)(\gamma^2 a^2 + 3\gamma a + 3)\Big], \tag{3.13}$$

where we have used (2.14) and (3.9).

The forces F_1 and F_2 having been found with the use of (3.13) and (3.7), we can now determine μ_G and K near the sphere. When we equate (3.5) and (3.7), i.e. $F_3 = F_2$, we obtain the equation for the determination of μ_G, namely

$$\mu_G U_G = \mu_1(\varkappa_1 U_1 + \varkappa_2\mu^* U_2),$$

thus is

$$\mu_G = \frac{\mu_1(\varkappa_1 U_1 + \varkappa_2\mu^* U_2)}{\varkappa_1 U_1 + \varkappa_2 U_2}. \tag{3.14}$$

When we set $F_1 = F_3$ according to (3.13) and (3.5), we get the relation

$$(\varkappa_1 U_1 + \varkappa_2 U_2)(\varkappa_1 + \varkappa_2\mu^*) - (\varkappa_1 U_1 + \varkappa_2\mu^* U_2) = \frac{1}{3}\varkappa_1\varkappa_2(\mu^* - 1)(U_1 - U_2)(\gamma^2 a^2 + 3\gamma a + 3)$$

from which we can find the interaction coefficient K. Solving this equation we obtain

$$\gamma a = -3. \tag{3.15}$$

According to (2.5) we see that

$$\gamma^2 = K\left(\frac{1}{\varkappa_1\mu_1} + \frac{1}{\varkappa_2\mu_2}\right)$$

and (3.15) then supplies us with a formula for K, namely

$$K = \frac{9}{a^2} \cdot \frac{\varkappa_1\mu_1\varkappa_2\mu_2}{\varkappa_1\mu_1 + \varkappa_2\mu_2}. \tag{3.16}$$

When we put the obtained expressions of μ_G and K, i.e. (3.14) and (3.16), into (3.13), we see that (3.7) and (3.13) are equal and thus is $F_1 = F_2$.

The found results show that μ_G and K can be computed with the use of the known parameters of both components of the medium. For the determination of the pressure p of the two-component medium we have according to [3] the formulas

$$\frac{\partial p}{\partial r} = -\frac{\mu_1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \theta} (E^2 \chi_1),$$

$$\frac{\partial p}{\partial \theta} = \frac{\mu_1}{\sin \theta} \frac{\partial}{\partial r} (E^2 \chi_1).$$

We put in here $E^2\chi_1$ from (3.6) and get

$$\begin{aligned} dp &= \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial \theta} d\theta \\ &= -\frac{3}{2} aU^* \mu_1 \left(\frac{2 \cos \theta}{r^3} dr + \frac{\sin \theta}{r^2} d\theta \right) \\ &= \frac{3}{2} aU^* \mu_1 \, d \left(\frac{\cos \theta}{r^2} \right) \end{aligned}$$

from which follows

$$p = p_\infty + \frac{3}{2} \mu_1 a U^* \frac{\cos \theta}{r^2} \tag{3.17}$$

Here p_∞ is the pressure of the two-component medium far from the sphere.

4. The convective mass transport in a two-component flow round a solid sphere

Let us consider the diffusion process of a substance immersed in a two-component medium on a solid sphere. The appropriate hydrodynamic problem of the two-component flow round a solid sphere has been solved in the previous items of the paper. Starting from these results we can now deal with the solution of the diffusion problem.

We consider the following problem:

The two-component medium includes a little admixture which, in contact with the solid sphere, leads to chemical or physico-chemical changes. We search for the total diffusion stream to the solid spherical particle.

The concentration c of the admixture may be small so that the diffusion coefficient D can be assumed as constant. The Reynolds numbers Re_i have been supposed to be small. As we can see from the solution of the hydrodynamic problem the velocities v_θ and v_r decrease continuously with the distance from the particle surface and in the immediate neighbourhood of the solid sphere a hydrodynamic boundary layer does not exist. In spite of that however a diffusion boundary layer develops near the particle surface. It is connected with the fact that the appropriate Peclét numbers $\mathrm{Pe}_i = \dfrac{U_i a}{D}$

are many times greater than the Reynolds numbers Re_i. Thus there are $\mathrm{Re}_i \ll 1$ and $\mathrm{Pe}_i \gg 1$ simultaneously. As known for $\mathrm{Pe} \gg 1$ the essential diffusion processes take place quite near the reaction surface. Here in the diffusion boundary layer we can observe a clear change of the concentration of the admixture.

The equation of the convective diffusion in the diffusion boundary layer in spherical coordinates is given by

$$v_r \frac{\partial c}{\partial r} + \frac{v_\theta}{r} \frac{\partial c}{\partial \theta} = D \left(\frac{\partial^2 c}{\partial r^2} + \frac{2}{r} \frac{\partial c}{\partial r} \right). \tag{4.1}$$

Here on the right side the term $\dfrac{1}{r^2 \sin \theta} \dfrac{\partial}{\partial \theta} \left(\sin \theta \dfrac{\partial c}{\partial \theta} \right)$ is omitted, because the derivatives along the sphere surface compared with the derivatives with respect to the radius vector are small. We can take the velocities v_r and v_θ according to (2.20).

For solving (4.1) we still need the boundary conditions. Since the diffusion process takes place in the immediate neighbourhood of the solid spherical particle, the condition far from the sphere is

$$c = c_0 \quad \text{for} \quad r \to \infty. \tag{4.2}$$

Now we consider such a diffusion process for which the condition

$$c = 0 \quad \text{for} \quad r = a \tag{4.3}$$

is satisfied on the sphere surface. That means that the admixture contacted with the particles reacts immediately with them and therefore the greatest possible diffusion stream is guaranteed. At last we assume that in the stagnation point singularities do not exist, i.e. it is

$$c = c_0 \quad \text{in the stagnation point.} \tag{4.4}$$

The problem now consists in solving the equation (4.1) with the boundary conditions (4.2)—(4.4) and in the determination of an expression of the total diffusion stream.

On account of the equation of continuity it is

$$v_r = -\frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, \quad v_\theta = \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}, \tag{4.5}$$

where ψ is the stream function,

$$\psi = \frac{1}{\varkappa_1 + \varkappa_2 \mu^*} [\chi_1 + \varkappa_1 \varkappa_2 (\mu^* - 1)\, \chi_2], \tag{4.6}$$

and χ_1, χ_2 can be found according to (2.16) and (2.18). Because the essential diffusion processes take place in the diffusion boundary layer, only solutions of the equation (4.1) for such values of r are interesting for us, which differ little from the sphere radius a. We set $x = r - a$. In view of (2.16) and (2.18) we then obtain for small x

$$\chi_1 = \frac{3}{4} U^* \sin^2 \theta\, x^2, \quad \chi_2 = \frac{3}{4} \bar{U} \sin^2 \theta\, (\gamma a + 1)\, x^2, \tag{4.7}$$

and according to (4.6) the stream function ψ is given by

$$\psi = \frac{3}{4} \sin^2 \theta \, x^2[\alpha U^* + \beta \overline{U}(\gamma a + 1)] \tag{4.8}$$

where

$$\alpha = \frac{1}{\varkappa_1 + \varkappa_2 \mu^*}, \quad \beta = \frac{\varkappa_1 \varkappa_2 (\mu^* - 1)}{\varkappa_1 + \varkappa_2 \mu^*}.$$

When we change the variables θ, r of the equation (4.1) to the variables θ, ψ and we regard $\frac{\partial^2 c}{\partial x^2} \gg \frac{2}{a} \frac{\partial c}{\partial x}$ for $x \ll a$, we obtain the following equation at the diffusion boundary layer:

$$\frac{\partial c}{\partial \theta} = Da^2 \sin^2 \theta \frac{\partial}{\partial \psi} \left(a v_\theta \frac{\partial c}{\partial \psi} \right). \tag{4.9}$$

From (4.5) and (4.8) can be found

$$v_\theta = \frac{3}{2} x \frac{\sin \theta}{a} [\alpha U^* + \beta \overline{U}(\gamma a + 1)]. \tag{4.10}$$

Putting (4.10) into (4.9) and carrying out the substitution

$$\xi = Da^2 \sqrt{3[\alpha U^* + \beta \overline{U}(\gamma a + 1)]} \int \sin^2 \theta \, d\theta \tag{4.11}$$

we get the equation

$$\frac{\partial c}{\partial \xi} = \frac{\partial}{\partial \psi} \left(\sqrt{\psi} \frac{\partial c}{\partial \psi} \right). \tag{4.12}$$

Instead of (4.2)—(4.4) we now have the following boundary conditions:

On the particle surface

$$c = 0 \quad \text{for} \quad \psi = 0; \tag{4.13}$$

far from the particle

$$c = c_0 \quad \text{for} \quad \psi \to \infty; \tag{4.14}$$

at the stagnation point

$$c = c_0 \quad \text{for} \quad \theta = 0, \quad \psi = 0. \tag{4.15}$$

The solution of the equation (4.12) in view of the boundary conditions (4.13)—(4.15) was found in [5]. Therefore we can write at once

$$c = \frac{c_0}{1{,}15} \int_0^\eta \exp\left(-\frac{4}{9} \eta^3\right) d\eta \tag{4.16}$$

where

$$\eta = \sqrt[3]{\frac{3[\alpha U^* + \beta \overline{U}(\gamma a + 1)}{4Da^2}} \cdot \frac{x \sin \theta}{\left(\theta - \frac{\sin 2\theta}{2}\right)^{1/3}}. \tag{4.17}$$

The diffusion stream on the particle surface can be determined by

$$j = D \left. \frac{\partial c}{\partial x} \right|_{x=0} .$$

According to (4.16) and (4.17) we obtain

$$j = D \frac{c_0}{1{,}15} \sqrt[3]{\frac{3[\alpha U^* + \beta \bar{U}(\gamma a + 1)]}{4Da^2}} \frac{\sin \theta}{\left(\theta - \frac{\sin 2\theta}{2}\right)^{1/3}} . \tag{4.18}$$

The thickness of the diffusion boundary layer can be computed by

$$\delta = \frac{Dc_0}{j} = 1{,}15 \sqrt[3]{\frac{4Da^2}{3[\alpha U^* + \beta \bar{U}(\gamma a + 1)]}} \frac{\left(\theta - \frac{\sin 2\theta}{2}\right)^{1/3}}{\sin \theta} . \tag{4.19}$$

The total diffusion of the admixture on the solid sphere can be found by

$$J = \int j \, ds = 2\pi a^2 \int_0^\pi j \sin \theta \, d\theta .$$

In view of (4.18) we then obtain

$$J = 7{,}98 \cdot c_0 D^{2/3} a^{4/3} [\alpha U^* + \beta \bar{U}(\gamma a + 1)]^{1/3} . \tag{4.20}$$

This formula includes the interaction coefficient K. In the item 3 we have determined K in the immediate neighbourhood of the particle. When we set, according to (3.16), the obtained expression for K into (4.20), we then get

$$J = 7{,}98 \cdot c_0 D^{2/3} a^{4/3} \left\{ \frac{1}{\varkappa_1 + \varkappa_2 \mu^*} [\varkappa_1 U_1 + \varkappa_2 \mu^* U_2 - 2\varkappa_1 \varkappa_2 (\mu^* - 1)(U_1 - U_2)] \right\}^{1/3} \tag{4.21}$$

in dependence on the parameters of the two-component medium. The function of θ from the formula (4.18) has, at the point $\theta = 0$, the value 1, at the point $\theta = \frac{\pi}{2}$ the value $\sqrt[3]{\frac{2}{\pi}}$ and for $\theta = \pi$ the value 0. Thus the diffusion stream has the greatest value at the stagnation point $\theta = 0$, it decreases with increasing θ.

The thickness δ of the diffusion boundary layer (4.19) becomes greater with increasing θ and goes to infinity for $\theta = \pi$. At the beginning we have assumed the thickness of the diffusion boundary layer to be much smaller than the particle radius. Thus we can conclude that beginning from some values θ placed near $\theta = \pi$, the considered theory is no longer applicable. Apart from that the domain $\theta \sim \pi$ however has little influence on the total stream J of the admixture.

In the end it may be remarked that the diffusion stream j, the thickness of the diffusion boundary layer δ and the total diffusion stream J of the admixture on the particle depend on the parameters of the components of the two-component medium. Already for a quasihomogeneous flow, i.e. when $U_1 = U_2$, the obtained formulas (4.18), (4.19) and (4.21) are converted into the formulas of a homogeneous flow [5].

In other words, when the velocities of both the components are equal far from the solid sphere, the interaction between the components is not important for solving the diffusion problem and we can immediately take the known formulas for a homogeneous medium.

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